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RIGIDITY IN THE HARMONIC MAP HEAT FLOW

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Abstract

We establish various uniformity properties of the harmonic map heat flow, including uniform convergence in L^2 exponentially as $t \to \infty$, and uniqueness of the positions of bubbles at infinite time. Our hypotheses are that the flow is between 2-spheres, and that the limit map and any bubbles share the same orientation.

1. Introduction

Let us consider smooth maps $\phi : S^2 \to S^2$. We use $z = x + iy$ as a complex coordinate on the domain, obtained by stereographic projection, and write the metric as $\sigma^2 dz d\bar{z}$, where

$$\sigma(z) = \frac{2}{1 + |z|^2}.$$

Similarly we have a coordinate u on the target, and a metric $\rho^2 du d\bar{u}$. We are using the notation

$$dz = dx + idy, \qquad d\bar{z} = dx - idy,$$

with analogues for du and $d\bar{u}$, and we will write

$$u_z = \frac{1}{2}(u_x - iu_y), \qquad u_{\bar{z}} = \frac{1}{2}(u_x + iu_y).$$

To the map ϕ we associate the energy densities

$$e_\partial(\phi) = \frac{\rho^2(u)}{\sigma^2}|u_z|^2, \qquad e_{\bar{\partial}}(\phi) = \frac{\rho^2(u)}{\sigma^2}|u_{\bar{z}}|^2,$$

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and

$$e(\phi) = e_{\partial}(\phi) + e_{\bar{\partial}}(\phi).$$

The corresponding energies are

$$E_{\partial}(\phi) = \int_{S^2} e_{\partial}(\phi) = \frac{i}{2}\int_{\mathbb{C}} \rho^2 |u_z|^2 dz \wedge d\bar{z},$$

$$E_{\bar{\partial}}(\phi) = \int_{S^2} e_{\bar{\partial}}(\phi) = \frac{i}{2}\int_{\mathbb{C}} \rho^2 |u_{\bar{z}}|^2 dz \wedge d\bar{z},$$

and

$$E(\phi) = E_{\partial}(\phi) + E_{\bar{\partial}}(\phi). \tag{1}$$

We also define a local energy

$$E_{(x,r)}(\phi) = \int_{\mathbb{B}_r(x)} e(\phi),$$

where $\mathbb{B}_r(x)$ is the geodesic ball of radius r centred at x in S^2. The Jacobian of ϕ is given by

$$J(\phi) = e_{\partial}(\phi) - e_{\bar{\partial}}(\phi),$$

and consequently we see that

$$E_{\partial}(\phi) - E_{\bar{\partial}}(\phi) = 4\pi deg(\phi). \tag{2}$$

For a fixed target chart, we may form

$$\tau = \frac{4}{\sigma^2}\left(u_{z\bar{z}} + \frac{2\rho_u}{\rho} u_z u_{\bar{z}}\right) \tag{3}$$

The associated geometric object is $\tau\frac{\partial}{\partial u}$, and the tension of ϕ is defined to be

$$\mathcal{T}(\phi) = \tau\frac{\partial}{\partial u} + \bar{\tau}\frac{\partial}{\partial \bar{u}},$$

which is a section of $\phi^*(TS^2)$.

The critical points of the energy functional E are known as harmonic maps, and the Euler-Lagrange equation which they satisfy is

$$\mathcal{T}(\phi) = 0.$$

The harmonic map heat flow is a solution $\Phi : S^2 \times [0, \infty) \to S^2$ of the associated parabolic problem

$$\frac{\partial \Phi}{\partial t} = \mathcal{T}(\Phi), \qquad \Phi(\cdot, 0) = \phi_0, \tag{4}$$

which we refer to as the 'heat equation'. We call ϕ_0 the 'initial map'. The heat flow was introduced by Eells and Sampson [2]. It is L^2-gradient flow on the energy - loosely speaking, Φ evolves in order to decrease its energy as quickly as possible.

For $\Omega \subset\subset S^2$ we measure the concentration over Ω of a flow with the quantity

$$\mathcal{E}(R, \Omega) = \sup_{(x,t) \in \Omega \times [0,\infty)} E_{(x,R)}(\Phi(\cdot, t)).$$

We will have cause to embed the target S^2 in $\mathbb{R}^3$, and see Φ as a map $v : \mathbb{R}^2 \times [0, \infty) \to S^2 \hookrightarrow \mathbb{R}^3$, or $\hat{v} : S^2 \times [0, \infty) \to S^2 \hookrightarrow \mathbb{R}^3$, and ϕ_0 as a map $\hat{v}_0 : S^2 \to S^2 \hookrightarrow \mathbb{R}^3$. In terms of v we have

$$e(\phi) = \frac{1}{2} |\nabla v|^2,$$

and so

$$E(\phi) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla v|^2 dx dy.$$

In the harmonic map heat flow, the map v evolves according to

$$\frac{\partial v}{\partial t} = \frac{1}{\sigma^2} (\Delta v + v |\nabla v|^2), \tag{5}$$

where Δ is the Laplace-Beltrami operator. We will denote the right-hand side of (5) also by $\mathcal{T}$, without confusion. We may also make references such as $E(v(\cdot, t))$ meaning $E(\Phi(\cdot, t))$ for related v and Φ.

Existence theory for the heat equation was studied by Struwe in [6]. The equation was shown to have a global weak solution (which is now known to be essentially unique [3]) and to be smooth except at finitely many points in space-time. For much of this paper we will be considering only the asymptotic behaviour of the flow at infinite time, in which case we may assume it is globally smooth.

Struwe's theory also described how 'bubbles' may occur in the heat flow. We need an extension of this which is the following theorem of Qing [4].

Theorem 1. *Let v be the solution of* (5), *corresponding to a solution Φ of* (4). *Then there exist finitely many non-constant harmonic maps $\{\hat{\omega}_k\}_{k=0}^m$ from S^2 to S^2 (seen as maps $\{\omega_k\}_{k=0}^m$ from $\mathbb{R}^2$ to S^2 via stereographic projection) together with sequences*

(i) $\{t_i\}$ *with* $t_i \to \infty$,

(ii) $\{\{a_i^k\}\}_{k=1}^m$ *in* $\mathbb{R}^2$ *with* $\lim_{i\to\infty} a_i^k = x^k \in \mathbb{R}^2$ *for* $1 \leq k \leq m$ *(where x^k corresponds to a point $\hat{x}^k \in S^2$), and*

(iii) $\{\{\lambda_i^k\}\}_{k=1}^m$ *with* $\lambda_i^k > 0$ *for* $1 \leq k \leq m$ *and any i, and* $\lim_{i\to\infty} \lambda_i^k = 0$ *for* $1 \leq k \leq m$,

such that

$$\frac{\lambda_i^k}{\lambda_i^j} + \frac{\lambda_i^j}{\lambda_i^k} + \frac{|a_i^k - a_i^j|^2}{\lambda_i^k \lambda_i^j} \to \infty, \qquad \text{as } i \to \infty,$$

and

$$\lim_{t\to\infty} E(v(\cdot,t)) = \sum_{k=0}^{m} E(\omega_k), \tag{6}$$

and moreover,

$$v(x,t_i) - \sum_{k=1}^{m} \left(\omega_k \left(\frac{x - a_i^k}{\lambda_i^k} \right) - \omega_k(\infty) \right) \to \omega_0$$

strongly in $W^{1,2}_{loc}(\mathbb{R}^2, \mathbb{R}^3)$ *as* $i \to \infty$.

Remark 1. In fact, Qing proves more than Theorem 1. We are using his theorem on Palais-Smale-type sequences rather than his theorem on the harmonic map heat flow. Of course, to be able to do this we must find a sequence $\{t_i\}$ with $t_i \to \infty$ such that $\mathcal{T}(\Phi(\cdot,t_i)) \to 0$ in $L^2(S^2)$. Such a sequence is easy to find (see [6] or deduce it from (16).) Qing's work has now been generalised by Ding and Tian [1].

We will refer to the map $\hat{\omega}_0$ as the 'body map', and to the maps $\{\hat{\omega}_k\}_{k=1}^m$ as 'bubbles'. The points $\{\hat{x}^k\}_{k=1}^m$ will be known as bubble points, or blow-up points.

Of course, for (6) to hold, we should choose our domain chart so that none of the blow-up points correspond to the point at infinity in the domain, though this is just a technical point.

As the statement of Qing's theorem is technical, we will describe some of its implications in more intuitive language. The theorem says firstly that there exists a sequence of times $\{t_i\}$ at which the heat flow $\hat{v}$ converges weakly in $W^{1,2}(S^2, \mathbb{R}^3)$ to the harmonic map $\hat{\omega}_0$ (and in particular strongly in $L^p(S^2, \mathbb{R}^3)$ for $p \in [1, \infty)$) and that we have the strong convergence

$$\hat{v}(\cdot, t_i) \to \hat{\omega}_0 \quad \text{in} \quad W^{1,2}_{loc}(S^2 \backslash \{x^1 \dots x^m\}, \mathbb{R}^3) \quad \text{as} \quad i \to \infty. \tag{7}$$

The theorem also says that near the bubble points $\{x^k\}_{k=1}^m$, the energy of the flow concentrates, and that by rescaling appropriate regions by appropriate amounts, we see new maps - the bubbles. So much was known from the work of Struwe [6]. An important aspect of the theorem is that it tells us that all the energy of the flow is accounted for by the body map and the bubbles.

In this paper we show that some of the asymptotic properties of the heat flow hold uniformly as $t \to \infty$ rather than just at a special sequence of times $\{t_i\}$.

2. Statement of the results

Before we state our results, we must recall that any harmonic map from S^2 to S^2 (or more generally from S^2 to a surface) is either holomorphic or anti-holomorphic. The proof is simple and may be found in [8] for example.

We now give our main theorem.

Theorem 2. *Suppose we have a solution Φ of the heat equation (4), and the corresponding v and $\hat{v}$. Suppose moreover that at infinite time, the bubbles and the body map are all holomorphic or all anti-holomorphic. Then with the definition of $\hat{\omega}_0$ and $\{\hat{x}^k\}_{k=1}^m$ as in Theorem 1 we have that:*

- *(i) $\hat{v}(\cdot, t) \rightharpoonup \hat{\omega}_0$ uniformly as $t \to \infty$ weakly in $W^{1,2}(S^2, \mathbb{R}^3)$ - and hence strongly in $L^p(S^2, \mathbb{R}^3)$ for any $p \in [1, \infty)$,*
- *(ii) $\hat{v}(\cdot, t) \to \hat{\omega}_0$ uniformly as $t \to \infty$ in $C^k_{loc}(S^2 \backslash \{\hat{x}^1 \dots \hat{x}^m\}, \mathbb{R}^3)$ for any $k \in \mathbb{N}$,*
- *(iii) for any $r > 0$ sufficiently small and $k \in \{1 \dots m\}$, the quantity $E_{(\hat{x}^k, r)}(\Phi(\cdot, t))$ converges to a limit $F_{k,r}$ uniformly as $t \to \infty$.*

Remark 2. In fact, we can control the *rate* of convergence too (see [7]). It will follow from the proofs of our theorems, for example, that given $\Omega \subset\subset S^2\backslash\{\hat{x}^1 \ldots \hat{x}^m\}$ there exist $C > 0$ and $\gamma > 0$ such that:

(i) $\|\hat{v}(\cdot,t) - \hat{\omega}_0\|_{L^2(S^2)} \leq C\left(E_\partial(\hat{v}(\cdot,t))\right)^{\frac{1}{2}} \leq Ce^{-\gamma t}$,

(ii) $\|\hat{v}(\cdot,t) - \hat{\omega}_0\|_{W^{1,2}(\Omega)} \leq C\left(E_\partial(\hat{v}(\cdot,t))\right)^{\frac{1}{2}} \leq Ce^{-\gamma t}$,

(iii) $|E_{(\hat{x}^k,r)}(\Phi(\cdot,t)) - F_{k,r}| \leq C\left(E_\partial(\hat{v}(\cdot,t))\right)^{\frac{1}{2}} \leq Ce^{-\gamma t}$.

A consequence of parts (ii) and (iii) of Theorem 2 is that we cannot pick another sequence of times $\{t_i\}$ in Qing's theorem and get a different set of blow-up points $\{\hat{x}^k\}$.

Although our result may be true if we allow some of the maps $\{\omega_k\}_{k=0}^m$ to be holomorphic and others anti-holomorphic (in other words absolutely no restrictions on the flow) it is no longer true if we drop the condition that the target is S^2. An example in which parts (i) and (ii) fail is given later. In fact, we believe that part (iii) may also fail - a sketched counter-example will be given in [7] in which a different sequence of times $\{t_i\}$ gives a different number of bubbles.

We remark that we do have an example of a heat flow satisfying the hypotheses of Theorem 2 in which bubbling occurs. We will give this in [7]. The body map in this example is constant.

We also have the following perturbation result for flows which are smooth for all time (i.e., flows with no bubbles at finite time - see Struwe [6].)

Theorem 3. *Suppose we have two solutions of the heat equation, which we write as maps $\hat{v}$ and $\hat{w}$ from $S^2 \times [0,\infty)$ to $S^2 \hookrightarrow \mathbb{R}^3$, with initial maps $\hat{v}_0$ and $\hat{w}_0$ from S^2 to $S^2 \hookrightarrow \mathbb{R}^3$. Suppose moreover that for the flow $\hat{v}$ there are no bubbles at finite time and that the bubbles and the body map at infinite time are all holomorphic or all anti-holomorphic.*

Then with $\{\hat{x}^k\}$ the blow-up points of the flow $\hat{v}$ as in Theorem 1, we have that for all $\varepsilon > 0$, $\Omega \subset\subset S^2\backslash\{\hat{x}^1 \ldots \hat{x}^m\}$ and $r > 0$ sufficiently small, there exists $\delta > 0$ independent of $\hat{w}$ such that if

$$\|\hat{v}_0 - \hat{w}_0\|_{W^{1,2}(S^2)} < \delta,$$

then

(i) $\|\hat{v}(\cdot,t) - \hat{w}(\cdot,t)\|_{L^2(S^2)} < \varepsilon$ *for all* $t > 0$,

(ii) $\|\hat{v}(\cdot,t) - \hat{w}(\cdot,t)\|_{W^{1,2}(\Omega)} < \varepsilon$ *for all* $t > 0$,

In other words, if we start a new flow close to the original one, then it will stay close in L^2 for all time, and close in $W^{1,2}$ away from the blow-up points. We remark that the perturbed flow may blow up in finite time.

As in Theorem 2 the hypothesis that the target is S^2, rather than something higher dimensional, cannot simply be dropped. In fact in [7] we will give an example of a harmonic map (from S^2 to a higher dimensional target) which blows up under arbitrarily small C^∞ perturbations. By considering a perturbation of a harmonic mapping from T^2 to an equator of S^2, we see that the theorem is not true for general domain surfaces.

Remark 3. The condition that the body map and bubbles at infinite time are all holomorphic or all anti-holomorphic will certainly be satisfied if we impose the condition

$$E(\phi_0) \leq 8\pi + 4\pi|deg(\phi_0)|$$

on the initial map ϕ_0. This follows from (1) and (2) and the fact that if $\phi : S^2 \to S^2$ is a non-trivial holomorphic map, then $E_{\partial}(\phi) \geq 4\pi$. We omit any further details.

3. The key estimate

In this section we derive a key estimate controlling the ∂-energy in terms of the tension. The estimate is very similar to the key estimate of Leon Simon in his important paper [5]. However, in the special case of maps between 2-spheres, and with the harmonic map energy functional E, we are able to reduce Simon's hypothesis of $W^{2,2}$ closeness to a harmonic map, to just smallness of the ∂-energy. This makes it applicable to maps with bubbles, assuming the bubbles and the body map are all anti-holomorphic (or all holomorphic).

Very loosely speaking, the harmonic map heat flow can only keep moving energy about for all time if the total energy is dissipated very slowly. The point of the estimate will be to show that this cannot happen, and thus that the heat flow becomes 'rigid' for large times.

Lemma 1. *There exist $\varepsilon_0 > 0$ and $\kappa > 0$ such that providing $\phi : S^2 \to S^2$ satisfies $E_{\partial}(\phi) < \varepsilon_0$, we have the estimate*

$$E_{\partial}(\phi) \leq \kappa \|\mathcal{T}(\phi)\|^2_{L^2(S^2)}. \tag{8}$$

Before proving Lemma 1 we recall the following lemma from [9, Theorem 2.8.4].

Lemma 2. *Suppose we have an operator I on functions $f:\mathbb{C}\to\mathbb{R}$ given by*

$$(If)(w)=\int_{\mathbb{C}}\frac{f(z)}{|z-w|}\frac{i}{2}dz\wedge d\bar{z}.$$

Then for $q\in(1,2)$ we have the estimate

$$\|If\|_{L^{\frac{2q}{2-q}}(\mathbb{C})}\leq C(q)\|f\|_{L^q(\mathbb{C})}.$$

Proof. (Lemma 1). Fix global complex coordinates z and u on the domain and target respectively. With these coordinates, we consider the quantity ρ^2u_z. To begin with, we calculate

$$(\rho^2u_z)_{\bar{z}} = \rho^2u_{z\bar{z}}+2\rho\rho_u u_{\bar{z}}u_z+2\rho\rho_{\bar{u}}\bar{u}_{\bar{z}}u_z \tag{9}$$

$$= \tfrac{1}{4}\sigma^2\rho^2\tau+2\rho\rho_{\bar{u}}|u_z|^2, \tag{10}$$

by (3). In particular, as $\rho_{\bar{u}}=-\frac{1}{2}u\rho^2$, we see that

$$|(\rho^2u_z)_{\bar{z}}|\leq|\sigma^2\rho\tau|+\rho^2|u_z|^2. \tag{11}$$

We now apply Cauchy's theorem for C^∞ functions to the function ρ^2u_z to get, for $|w|<r$,

$$\rho^2u_z(w)=\frac{1}{2\pi i}\int_{\partial D_r}\frac{\rho^2u_z}{z-w}dz+\frac{1}{2\pi i}\int_{D_r}\frac{(\rho^2u_z)_{\bar{z}}}{z-w}dz\wedge d\bar{z},$$

where $D_r=\{z\in\mathbb{C}\ :\ |z|<r\}$. Allowing r to tend to infinity, and observing that $|\rho^2u_z|\to 0$ as $|z|\to\infty$ (because $|\frac{\rho}{\sigma}u_z|$ is bounded) we see that

$$\rho^2u_z(w)=\frac{1}{2\pi i}\int_{\mathbb{C}}\frac{(\rho^2u_z)_{\bar{z}}}{z-w}dz\wedge d\bar{z}.$$

Combining this with (11), we have

$$|\rho^2u_z(w)|\leq\frac{1}{\pi}\int_{\mathbb{C}}\frac{1}{|z-w|}(|\sigma^2\rho\tau|+\rho^2|u_z|^2)\frac{i}{2}dz\wedge d\bar{z}.$$

Now we observe that the right-hand side is independent of which stereographic chart we took for the target (note for example that $|\rho\tau|$ is the length of $\tau\frac{\partial}{\partial u}$) so by taking a chart for which $0 \in \mathbb{C}$ corresponds to $\phi(w)$, we obtain the estimate

$$|\rho u_z(w)| \leq \frac{1}{2\pi}\int_{\mathbb{C}} \frac{1}{|z-w|}(|\sigma^2\rho\tau| + \rho^2|u_z|^2)\frac{i}{2}dz \wedge d\bar{z}. \tag{12}$$

But now, all the terms are independent of the target chart, allowing us to change target chart, or equivalently to move w with a fixed chart.

To develop estimate (12) we need to appeal to the theory of Riesz potentials, and in particular to Lemma 2. This produces, for $q \in (1,2)$, the first of the estimates

$$\|\rho u_z\|_{L^{\frac{2q}{2-q}}(\mathbb{C})} \leq C\left(\|\sigma^2\rho\tau\|_{L^q(\mathbb{C})} + \|\rho^2|u_z|^2\|_{L^q(\mathbb{C})}\right) \tag{13}$$

$$\leq C\left(\|\rho\tau\|_{L^q(S^2)} + \|\rho u_z\|^2_{L^{2q}(\mathbb{C})}\right) \tag{14}$$

$$\leq C\left(\|\rho\tau\|_{L^2(S^2)} + \|\rho u_z\|_{L^2(\mathbb{C})}\|\rho u_z\|_{L^{\frac{2q}{2-q}}(\mathbb{C})}\right), \tag{15}$$

whilst the last follows from applying Hölder's inequality to both terms. The constant C is changing, of course, but remains independent of ϕ. Now, as $E_\partial(\phi) = \|\rho u_z\|^2_{L^2(\mathbb{C})}$, we see that there exists $\varepsilon_0 > 0$ such that providing $E_\partial(\phi) \leq \varepsilon_0$, we may absorb the second term on the right-hand side into the left-hand side to give

$$\|\rho u_z\|_{L^{\frac{2q}{2-q}}(\mathbb{C})} \leq C\|\mathcal{T}\|_{L^2(S^2)}.$$

This then easily yields

$$\left(\int_{S^2_+} e_\partial(\phi)\right)^{\frac{1}{2}} = \|\rho u_z\|_{L^2(D_1)} \leq C\|\rho u_z\|_{L^{\frac{2q}{2-q}}(D_1)} \leq C\|\mathcal{T}\|_{L^2(S^2)},$$

where S^2_+ is the hemisphere corresponding to the points in the domain with $|z| < 1$. Repeating the estimate with the 'opposite' chart to give a ∂-energy estimate over the remaining hemisphere, and combining the two, we are left with

$$E_\partial(\phi) \leq C\|\mathcal{T}\|^2_{L^2(S^2)}.$$

q.e.d.

Remark 4. We remark that we have in fact proved more than stated in that we can control the L^p norm of $e_\partial(\phi)$ for any $p \geq 1$ not just $p = 1$.

Remark 5. Our key lemma is the part of this work which requires the hypotheses on the domain, target and flow. As we shall see, for a flow $\Phi(\cdot, t)$ the ∂-energy $E_\partial(\Phi(\cdot, t))$ is exactly half the energy still left to be dissipated during the flow, but in fact whenever we have an estimate of the form

$$(\text{energy left to dissipate}) \leq \|\mathcal{T}\|_{L^2(S^2)}^p$$

for some $p > 1$, the forthcoming proofs will be valid. Of course, such an estimate will not be true in general as the conclusions of our theorems are not true in general.

Although we only discuss the case of round 2-spheres in this work, we mention that the key lemma as stated implies the same lemma with a deformed domain metric, and that the proof can be modified to imply the same lemma with a deformed target metric.

4. Proof of the results

Before giving the proofs we state a supporting lemma which is a consequence of successive iterations of a result of Struwe [6, Lemma 3.10′]. The lemma gives control of C^k norms of the flow away from any points where the energy density concentrates.

Lemma 3. *There exists $\varepsilon_1 > 0$ such that whenever we have a solution $\Phi : S^2 \times [0, \infty) \to S^2$ of the heat equation* (4) *satisfying $\mathcal{E}(R, \Omega) < \varepsilon_1$ for some $R > 0$ and $\Omega \subset\subset S^2$, then the Hölder norms of Φ are bounded uniformly on $\Omega \times [\tau, \infty)$ for any $\tau > 0$.*

Proof. (Theorem 2). Without loss of generality, we will assume that the body map and all the bubbles are anti-holomorphic, rather than holomorphic.

We begin by recalling the well known fact that

$$\frac{d}{dt} E(\Phi(\cdot, t)) = -\|\mathcal{T}(\Phi(\cdot, t))\|_{L^2(S^2)}^2. \tag{16}$$

This is easily proved by writing E in terms of v, and using (5). Moreover, as a combination of (1) and (2) gives

$$E_\partial(\phi) = \frac{1}{2}\left(E(\phi) + 4\pi deg(\phi)\right),$$

we see that

$$\frac{d}{dt}E_\partial(\Phi(\cdot,t)) = -\frac{1}{2}\|\mathcal{T}(\Phi(\cdot,t))\|^2_{L^2(S^2)}. \tag{17}$$

Qing's description of the bubble tree in Theorem 1, together with the fact that the maps $\{\omega_k\}$ are all anti-holomorphic, tells us that at his sequence of times $\{t_i\}$, the ∂-energy is converging to zero. As the ∂-energy is decreasing (equation (17)) we then see that

$$E_\partial(\Phi(\cdot,t)) \to 0 \qquad \text{as} \qquad t \to \infty. \tag{18}$$

In particular, there exists a time T such that if $t \geq T$, then $E_\partial(\Phi(\cdot,t)) < \varepsilon_0$, where ε_0 is defined in Lemma 1. As we are concerned only with the asymptotics of the heat flow, we may suppose for simplicity that $T = 0$. Moreover, we may assume that no finite time blow-up occurs.

We note then that the combination of Lemma 1 and equation (17) implies the exponential decay of $E_\partial(\Phi(\cdot,t))$ which is necessary to establish the exponential convergence mentioned in Remark 2. An alternative application of Lemma 1 gives

$$\begin{aligned}-\frac{d}{dt}\left(E_\partial(\Phi(\cdot,t))\right)^{\frac{1}{2}} =& \frac{1}{4}\left(E_\partial(\Phi(\cdot,t))\right)^{-\frac{1}{2}}\|\mathcal{T}(\Phi(\cdot,t))\|^2_{L^2(S^2)}\\ \geq& \frac{1}{4\sqrt{\kappa}}\|\mathcal{T}(\Phi(\cdot,t))\|_{L^2(S^2)},\end{aligned}$$

and thus, for $t_0 \in [0,\infty)$

$$\int_{t_0}^{\infty}\|\mathcal{T}(\Phi(\cdot,t))\|_{L^2(S^2)}dt \leq C\left(E_\partial(\Phi(\cdot,t_0))\right)^{\frac{1}{2}}. \tag{19}$$

The first application of (19) is the calculation

$$\begin{aligned}\sup_{t\in[t_0,\infty)}\|\hat{v}(\cdot,t)-\hat{\omega}_0\|_{L^2(S^2)} \leq& \int_{t_0}^{\infty}\left\|\frac{\partial\hat{v}}{\partial t}\right\|_{L^2(S^2)}dt\\ =& \int_{t_0}^{\infty}\|\mathcal{T}(\Phi(\cdot,t))\|_{L^2(S^2)}dt\\ \leq& C\left(E_\partial(\Phi(\cdot,t_0))\right)^{\frac{1}{2}}.\end{aligned} \tag{20}$$

The exponential decay of $E_\partial(\Phi(\cdot,t_0))$ then gives us the L^2 exponential convergence of Remark 2. In pursuit of Theorem 2, however, we are satisfied with the weaker statement

$$\hat{v}(\cdot,t) \to \hat{\omega}_0 \text{ in } L^2(S^2,\mathbb{R}^3) \text{ as } t \to \infty. \tag{21}$$

A consequence of this is that

$$\hat{v}(\cdot,t) \rightharpoonup \hat{\omega}_0 \text{ weakly in } W^{1,2}(S^2,\mathbb{R}^3) \text{ as } t \to \infty. \tag{22}$$

This is because otherwise we could pick a sequence of times $\{t_i\}$ to give

$$|\langle \hat{v}(\cdot,t_i),\alpha\rangle - \langle \hat{\omega}_0,\alpha\rangle| > \delta, \tag{23}$$

where α is some test function, $\delta > 0$, and $\langle\cdot,\cdot\rangle$ is the inner product of $W^{1,2}(S^2,\mathbb{R}^3)$. Then, as the total energy E is bounded ($E(\Phi(\cdot,t)) \le E(\Phi(\cdot,0))$) we could pass to a subsequence of times (also called $\{t_i\}$) such that

$$\hat{v}(\cdot,t_i) \rightharpoonup \beta \text{ weakly in } W^{1,2}(S^2,\mathbb{R}^3) \text{ as } i \to \infty, \tag{24}$$

for some β. The convergence would thus be strong in $L^2(S^2,\mathbb{R}^3)$, and so by (21) we would have $\beta = \hat{\omega}_0$. There would then be a contradiction between (23) and (24). So (22) holds, which is part (i) of Theorem 2.

Of course, as $W^{1,2}$ is compactly embedded in L^p for any $p \in [1,\infty)$, the convergence in (22) tells us that

$$\hat{v}(\cdot,t) \to \hat{\omega}_0 \text{ in } L^p(S^2,\mathbb{R}^3) \text{ as } t \to \infty.$$

We now proceed to consider the local oscillation of energy. For $r,s > 0$, define a cut-off function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ by

$$\varphi(x) = \begin{cases} 1 & \text{if } |x| \le r, \\ 1 + \frac{1}{s}(r - |x|) & \text{if } r < |x| < r+s, \\ 0 & \text{if } |x| \ge r+s, \end{cases}$$

and define the 'cut energy' Θ_w of a map $w : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$\Theta_w = \Theta_w^{(r,s)} = \frac{1}{2}\int_{\mathbb{R}^2} \varphi^2 |\nabla w|^2.$$

We also write

$$\Theta(t) = \Theta^{(r,s)}(t) = \Theta_{v(\cdot,t)}^{(r,s)}.$$

The cut energy is about the point in S^2 corresponding to the point $0 \in \mathbb{R}^2$, but this could be any point by taking a different chart. The energy Θ evolves according to

$$\frac{d\Theta}{dt} = \int_{\mathbb{R}^2} \varphi^2 \nabla v^i . \nabla \mathcal{T}^i = -\int_{\mathbb{R}^2} \varphi^2 |\mathcal{T}|^2 \sigma^2 - 2\int_{\mathbb{R}^2} \varphi (\nabla\varphi . \nabla v^i)\mathcal{T}^i.$$

Abandoning the first term on the right, and using Hölder's inequality, we estimate

$$\frac{d\Theta}{dt} \leq \frac{2}{s}\Theta^{\frac{1}{2}}\|\mathcal{T}\|_{L^2(S^2)},$$

where we are assuming that $r + s < 1$ to avoid an extra constant. Integrating this, we see that the cut energy can only vary within the restriction

$$\Theta^{\frac{1}{2}}(t) - \Theta^{\frac{1}{2}}(t_0) \leq \frac{1}{s}\int_{t_0}^{t} \|\mathcal{T}(\Phi(\cdot,\xi))\|_{L^2(S^2)} d\xi,$$

and thus, by (19),

$$\Theta^{\frac{1}{2}}(t) - \Theta^{\frac{1}{2}}(t_0) \leq \frac{C}{s}\left(E_\partial(\Phi(\cdot,t_0))\right)^{\frac{1}{2}}, \tag{25}$$

where $t > t_0$, of course.

The power of (25) is evident. It would be required to establish the exponential convergence of part (iii) of Remark 2. However, we will be using the simple consequence that there exists a number $l^{(r,s)}$ such that

$$\Theta^{(r,s)}(t) \to l^{(r,s)} \qquad \text{as} \qquad t \to \infty. \tag{26}$$

For any $\Omega \subset\subset S^2 \backslash \{\hat{x}^1 \ldots \hat{x}^m\}$ this provides us with the uniform control of concentration

$$\mathcal{E}(R,\Omega) < \varepsilon_1,$$

for some R, enabling us to apply Lemma 3 to get

$$\|\hat{v}(\cdot,t)\|_{C^k(\Omega)} \leq C(k) \text{ uniformly for } t \in [1,\infty),$$

for all k. We can deduce the convergence of a subsequence of any sequence $\hat{v}(\cdot,t_i)$ in $C^k(\Omega)$ for any k, and hence establish part (ii) of Theorem 2 via the obvious contradiction argument.

Having established part (ii), part (iii) of Theorem 2 then follows from a further application of (26). q.e.d.

Before proving Theorem 3, we need a perturbation result for finite time intervals.

Lemma 4. *Suppose we have two solutions of the heat equation, which we write as maps $\hat{v}$ and $\hat{w}$ from $S^2 \times [0,\infty)$ to $S^2 \hookrightarrow \mathbb{R}^3$, with initial maps $\hat{v}_0$ and $\hat{w}_0$ from S^2 to $S^2 \hookrightarrow \mathbb{R}^3$. Suppose moreover that $T > 0$ and that the flow $\hat{v}$ has no bubbles up to time $t = T$ (in other words $\hat{v}$ is smooth for $t \in [0,T]$.)*

Then for all $\varepsilon > 0$, there exists $\delta > 0$ independent of $\hat{w}$ such that if

$$\|\hat{v}_0 - \hat{w}_0\|_{W^{1,2}(S^2)} < \delta,$$

then

$$\|\hat{v}(\cdot, t) - \hat{w}(\cdot, t)\|_{W^{1,2}(S^2)} < \varepsilon \text{ for all } t \in [0, T].$$

Proof. (Lemma 4). We sketch the proof, which essentially follows from the work of Struwe.

For $\varepsilon_1 > 0$ we may choose R sufficiently small so that

$$\sup_{(x,t)\in S^2\times[0,T]} E_{(x,2R)}(\hat{v}(\cdot, t)) < \frac{\varepsilon_1}{4}.$$

This is possible as $\hat{v}$ is regular for $t \in [0, T]$. Then for $\|\hat{v}_0 - \hat{w}_0\|_{W^{1,2}(S^2)}$ sufficiently small, we may ensure that

$$\sup_{x\in S^2} E_{(x,2R)}(\hat{w}_0) < \frac{\varepsilon_1}{2}.$$

Thus by Struwe's local control on the increase of energy [6, Lemma 3.6], for $\eta > 0$ sufficiently small, we have that

$$\sup_{(x,t)\in S^2\times[0,\eta]} E_{(x,R)}(\hat{w}(\cdot, t)) < \varepsilon_1.$$

Here η is dependent on $\hat{v}$ only in terms of R, and essentially independent of $\hat{w}$. The lemma then follows for $T \leq \eta$ by [6, Remark 3.9]. By dividing up the interval $[0, T]$ into intervals of length no more than η and applying the lemma for $T \leq \eta$ iteratively, we establish the lemma for general, finite, T. q.e.d.

Proof. (Theorem 3). As in the proof of Theorem 2 we will assume without loss of generality that the body map and all the bubbles are anti-holomorphic, rather than holomorphic.

The basic idea of the proof is to use Lemma 4 to show that the flows stay close until the ∂-energy is small, and then use the techniques we developed in the proof of Theorem 2. We set ε_0 as in Lemma 1 in anticipation.

We will distinguish between the body maps of the flows $\hat{v}$ and $\hat{w}$ by calling them $\hat{v}_\infty$ and $\hat{w}_\infty$ respectively. In keeping with our previous notation, v and w will be the maps from $\mathbb{R}^2 \times [0, \infty)$ associated to $\hat{v}$ and $\hat{w}$ (the maps from $S^2 \times [0, \infty)$) via stereographic projection.

Part (i) of Theorem 3 will follow from (20). For any $\eta_1, \eta_2 > 0$, we may choose T sufficiently large so that

$$\|\hat{v}(\cdot,t) - \hat{v}_\infty\|_{L^2} < \eta_1 \qquad \text{for} \qquad t \geq T,$$

from part (i) of Theorem 2 and

$$E_\partial(\hat{v}(\cdot,t)) < \min(\eta_2, \frac{\varepsilon_0}{2}) \qquad \text{for} \qquad t \geq T,$$

from (18) and

$$\|\hat{v}(\cdot,t) - \hat{v}_\infty\|_{W^{1,2}(\Omega)} < \eta_3 \qquad \text{for} \qquad t \geq T,$$

from part (ii) of Theorem 2. Then for any $\eta_4 > 0$, we may apply Lemma 4 to find $\delta > 0$ such that providing $\|\hat{v}_0 - \hat{w}_0\|_{W^{1,2}(S^2)} < \delta$, we have

$$\|\hat{v}(\cdot,t) - \hat{w}(\cdot,t)\|_{W^{1,2}(S^2)} < \min(\eta_4, \frac{\varepsilon_0}{2}, \varepsilon) \tag{27}$$

for all $t \in [0,T]$. Therefore we must have

$$E_\partial(\hat{w}(\cdot,t)) \leq E_\partial(\hat{w}(\cdot,T)) < \min(\eta_2 + \eta_4, \varepsilon_0) \qquad \text{for} \qquad t \geq T,$$

and so we may use (20) to estimate

$$\|\hat{w}(\cdot,t) - \hat{w}_\infty\|_{L^2} < C(\eta_2 + \eta_4)^{\frac{1}{2}} \qquad \text{for} \qquad t \geq T.$$

Combining the above, we find that

$$\begin{aligned}\|\hat{v}(\cdot,t) - \hat{w}(\cdot,t)\|_{L^2} &\leq \|\hat{v}(\cdot,t) - \hat{v}_\infty\|_{L^2} + \|\hat{v}_\infty - \hat{v}(\cdot,T)\|_{L^2} \\ &\quad + \|\hat{v}(\cdot,T) - \hat{w}(\cdot,T)\|_{L^2} \\ &\quad + \|\hat{w}(\cdot,T) - \hat{w}_\infty\|_{L^2} + \|\hat{w}_\infty - \hat{w}(\cdot,t)\|_{L^2} \\ &< 2\eta_1 + \eta_4 + 2C(\eta_2 + \eta_4)^{\frac{1}{2}},\end{aligned}$$

for $t \geq T$, and thus by taking η_1, η_2, η_4 sufficiently small and using (27) again, we establish part (i) of Theorem 3.

To establish part (ii) we must control locally the oscillation of the first order part of the $W^{1,2}$ norm. By adapting the argument below, with $w = v$, we could establish the exponential convergence of part (ii) of Remark 2. With φ and Θ as in the proof of Theorem 2 we calculate

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}^2} \varphi^2 |\nabla(w(\cdot,t) - v_\infty)|^2\right) \\ &\qquad = \frac{d}{dt}\Theta_{w(\cdot,t)} - \frac{d}{dt}\left(\int_{\mathbb{R}^2} \varphi^2 \nabla w(\cdot,t).\nabla v_\infty\right).\end{aligned} \tag{28}$$

The second term on the right-hand side is controlled by

$$\left|\frac{d}{dt}\left(\int_{\mathbb{R}^2}\varphi^2\nabla w(\cdot,t).\nabla v_\infty\right)\right| = \left|\int_{\mathbb{R}^2}\varphi^2\nabla\mathcal{T}(w(\cdot,t)).\nabla v_\infty\right| \leq C(v_\infty,r,s)\|\mathcal{T}(w(\cdot,t))\|_{L^2(S^2)},$$

where we have integrated by parts and used Hölder's inequality. Together with (25) and (19) we now have enough information to integrate (28) giving

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^2}\varphi^2|\nabla(w(\cdot,t)-v_\infty)|^2-\frac{1}{2}\int_{\mathbb{R}^2}\varphi^2|\nabla(w(\cdot,T)-v_\infty)|^2\\
&\leq \Theta_{w(\cdot,t)}-\Theta_{w(\cdot,T)}+\int_T^t C(v_\infty,r,s)\|\mathcal{T}(w(\cdot,\xi))\|_{L^2(S^2)}d\xi\\
&\leq \left(\Theta^{\frac{1}{2}}_{w(\cdot,t)}+\Theta^{\frac{1}{2}}_{w(\cdot,T)}\right)\frac{C}{s}\left(E_\partial(w(\cdot,T))\right)^{\frac{1}{2}}+C(v_\infty,r,s)\left(E_\partial(w(\cdot,T))\right)^{\frac{1}{2}}\\
&\leq C\left(E_\partial(w(\cdot,T))\right)^{\frac{1}{2}}\leq C(\eta_2+\eta_4),
\end{aligned}$$

where the constant C on the final line is independent of the flow w assuming we insist that $\|\hat{v}_0-\hat{w}_0\|_{W^{1,2}(S^2)}<1$ (for example) so that we have a bound on the energy of $w(\cdot,t)$. Taking η_2 and η_4 sufficiently small, we may make the right-hand side as small as we desire. Consequently, for any $\eta_5>0$ we can ensure that

$$\|\hat{w}(\cdot,t)-\hat{v}_\infty\|_{W^{1,2}(\Omega)}-\|\hat{w}(\cdot,T)-\hat{v}_\infty\|_{W^{1,2}(\Omega)}\leq\eta_5.$$

Combining everything again, we see that

$$\begin{aligned}
\|\hat{w}(\cdot,t)-\hat{v}(\cdot,t)\|_{W^{1,2}(\Omega)} &\leq \|\hat{w}(\cdot,t)-\hat{v}_\infty\|_{W^{1,2}(\Omega)}+\|\hat{v}_\infty-\hat{v}(\cdot,t)\|_{W^{1,2}(\Omega)}\\
&\leq \eta_5+(\eta_3+\eta_4)+\eta_3,
\end{aligned}$$

for $t\geq T$, so by taking η_3,η_5 sufficiently small and making η_4 smaller if necessary (and using (27) again) we establish part (ii) of Theorem 3.

q.e.d.

5. An example of non-uniqueness

As promised earlier, we now give a counter-example to part (i) of Theorem 2 when the condition that the target is S^2 is dropped. No bubbling occurs. The flow has a 'winding' behaviour and has a circle of accumulation points. The example will also show that a perturbation of a locally energy minimising harmonic map may move far away under the heat flow. It therefore contrasts with the work of Leon Simon ([5]) in which these phenomena are ruled out under the hypothesis that the target is real analytic.

Let the domain be S^2 and the target $\mathbb{R}^2 \times S^2$. It is not important that $\mathbb{R}^2$ is non-compact - as we shall see, we are only concerned with a bounded region, so we could change it to a flat 2-torus. We give the domain the standard metric, but give the target a warped metric - if g and h are the standard metrics on $\mathbb{R}^2$ and S^2 respectively, then at a point $(z, x) \in \mathbb{R}^2 \times S^2$, we define the metric to be $g(z) + f(z)h(x)$ where $f : \mathbb{R}^2 \to \mathbb{R}_+$ is to be determined. In other words, the target is $\mathbb{R}^2 \times_f S^2$. We consider initial maps of the form $u_0(x) = (z_0, x)$ where z_0 is independent of x. Such maps give solutions of the heat equation (4) of the form $u(x,t) = (z(t), x)$, where $z : \mathbb{R}_+ \to \mathbb{R}^2$, and z evolves according to

$$\frac{dz}{dt} = -\nabla f(z(t)). \tag{29}$$

In other words, we have reduced the heat flow to finite-dimensional gradient flow for z on the function f. It remains to choose the function f so that z may not have a unique limit, and so that moving z an arbitrarily small amount from a point z_0 with $\nabla f(z_0) = 0$ (which corresponds to a harmonic map) will make the solution of (29) move away from z_0. To achieve this we take a 'downwardly spiralling gramophone record'

$$f(r,\theta) = \begin{cases} 1 & \text{if } r \leq 1, \\ 1 + e^{-\frac{1}{r-1}} \left(\sin(\frac{1}{r-1} + \theta) + 2\right) & \text{if } r > 1, \end{cases}$$

where (r, θ) are polar coordinates. Taking initial conditions with $r = 2$ say, the solution for z will spiral in to give the circle $r = 1$ as the accumulation set. Moreover, any point with $r = 1$ is a stationary point, but perturbing r to be slightly larger will make the solution of (29) spiral around, and move at least a distance 2 away.

Remark 6. Of course, we are not restricted to using S^2 in the example above. In particular, the same idea will work for domains of any dimension.

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